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THEOREM ON UNIFORM CONTINUITY OF NEWTON POTENTIAL

Abstract: *Newton Potential* $\int K_p(z;\zeta)d\gamma(\zeta)$ *is important in the theory of subharmonic and* δ*-subharmonic functions. Classical properties were presented in many monographs, for example, in the works of N. S. Landkoff and V. S. Azarin.* The paper considers the case: measure γ in the plane. For any $z \in \mathbb{C}$ we consider the Newton potential as an element of the spaces $L_p(\gamma;\mathbb{C})$. In this article we give a sufficient condition on a measure γ the function $\int K_p(z;\zeta)d\gamma(\zeta) \in L_p(\gamma;\mathbb{C})$ to be uniformly continious in the parameter z in $\mathbb{C}$.

Key words: *Newton potential, Borel measure, uniform continuity, Minkowski inequality, Lebesgue measure.*

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ТЕОРЕМА О РАВНОМЕРНОЙ НЕПРЕРЫВНОСТИ ПОТЕНЦИАЛА НЬЮТОНА

Анотация: Потенциал Ньютона $\int N_p(z;\zeta)d\gamma(\zeta)$ играет важную роль в теории субгармонических и delta-субгармонических функций. Классические свойства были представлены во многих монографиях. Отметим, что А.Ф. Гришин и А. Шуиги изучили случай, когда мера γ есть ограничение меры Лебега на полуплоскости. В работе рассматривается случай: мера γ в плоскости. Для любого $z \in \mathbb{C}$ мы рассматриваем потенциал Ньютона как элемент пространств $L_p(\gamma;\mathbb{C})$. В этой статье приводится достаточное условие на меру γ для того, чтобы функция $\int K_p(z;\zeta)d\gamma(\zeta) \in L_p(\gamma;\mathbb{C})$ была равномерно непрерывной по параметру z в $\mathbb{C}$.

Ключевые слова: потенциал Ньютона, борелевская мера, равномерная непрерывность, неравенство Минковского, мера Лебега.

The theory of generalized functions is a basis for progress in many areas of mathematics and finds extensive applications in physics. One of the subdivisions of mathematics which is strongly influenced by the theory of generalized functions is the theory of subharmonic functions. Using generalized functions Azarin [1] constructed a theory of limit sets of subharmonic functions and measures, which is a significant contribution to the theory of subharmonic functions.

We will use the following notation:

$$B(z_0;R)=\{z\in C:|z-z_0|\le R\};CB(z_0;R)=C/B(z_0;R).$$

In the theory of subharmonic and δ-subharmonic functions in the plane ℂ, an important role is played by the kernel:

$$N_p(z;\zeta)=\begin{cases}\operatorname{Re}\left(\ln\left(1-\frac{z}{\zeta}\right)+\frac{z}{\zeta}+\ldots+\frac{1}{p}\frac{z^p}{\zeta^p}\right),\\ \ln|z-\xi|\end{cases}$$

where $p\in\mathbb{N}$. For all $z,\zeta\in$ ℂ we have the inequality ([8], Lemma 2):

$$|N_p(z;\zeta)|\le M(p)\frac{|z|^p}{|\zeta|^p}\min\left\{1;\frac{|z|}{|\zeta|}\right\}, \tag{1}$$

where M(p) depends only on p. Let γ be the Radon measure in ℂ . We consider the following potential

$$N(z)=\int_C N_p(z;\zeta)d\gamma(\zeta),$$

which we shall call the Newton potential of the measure γ.

In this paper, we consider this Newton potential $N(z)$ as a map from the space ℂ into the space $L_p(C;d\gamma(\zeta))$. In this case, we can write $K(z):C\to L_p(\gamma)$.

In the present paper we find sufficient conditions on a measure γ with support compactly embedded in G that guarantee the convergence in the space $L_m(\gamma;C)$ of sequences of subharmonic functions that converge in the sense of the theory of generalized functions.

The uniform continuity of these functions is a restriction on the measure γ which appears in the theorems that are the principal results of the paper. It is important for what follows that the dependence of $N(z)$ be uniformly continuous.

Theorem. Let m > 1 be an arbitrary fixed number. Suppose that γ is a positive finite Borel measure with compact support that satisfies the condition supp $\gamma\subset$ ℂ. In addition, suppose that

$$\sup\left\{\int\limits_{B(z;\delta)} |N_p(z;\zeta)|^m \, d\gamma(\zeta) : z \in C\right\} \to 0; \tag{2}$$

$$(\delta \to 0).$$

Then the function $K(z): C \to L_m(\gamma)$ is uniformly continuous with respect to the variable z in the space ℂ.

Proof. We divide the proof into several stages.

1. Since we have

$$N_p(z;\zeta) = \mathrm{Re}\left(\ln\left(1-\frac{z}{\zeta}\right)\right) + \mathrm{Re}\left(\frac{z}{\zeta}+\ldots+\frac{1}{p}\frac{z^p}{\zeta^p}\right) = \ln\left|1-\frac{z}{\zeta}\right| + N_p^{\,1}(z;\zeta).$$

Obviously the function $K_p^{\,1}(z;\zeta): C \to L_m(\gamma)$ is uniformly continuous with respect to the variable z in the space ℂ.

2. We set

$$\psi(z_1;z_2) = \left(\int\limits_C |\ln|z_1-\zeta|-\ln|z_2-\zeta||^p \, d\gamma(\zeta)\right)^{1/p}; \quad F(z) = \left(\int\limits_C |\ln|z-\zeta||^p \, d\gamma(\zeta)\right)^{1/p}.$$

We do this at the first stage. It follows from the Minkowski inequality that

$$F(z) \le \left(\int\limits_{B(z;\delta)} |\ln|z-\zeta||^p \, d\gamma(\zeta)\right)^{1/p} + \left(\int\limits_{CB(z;\delta)} |\ln|z-\zeta||^p \, d\gamma(\zeta)\right)^{1/p} = J_1 + J_2.$$

where $CB(z;\delta) = C \setminus B(z;\delta)$.

We prove that the inequality

$$|\ln|z-\zeta|| \le \ln\frac{1}{\delta} \tag{3}$$

holds for sufficiently small δ and for $\zeta \in (\mathrm{supp}\,\gamma) \cap CB(z;\delta)$. From the inequality (3) we obtain that $J_2 \le \ln\frac{1}{\delta}(\gamma(C))^{1/p}$. By the hypothesis of the theorem, $J_1 \le 1$ for sufficiently small δ. Hence, $F(z)$ is finite. What has been proved can also be stated as follows: for any $z \in$ ℂ the function $K(z)$ is an element of the space $L_p(\gamma)$. The inequality $\psi(z_1;z_2) \le F(z_1) + F(z_2)$ implies that $\psi(z_1;z_2)$ is finite.

3. We prove that the inequality

$$\left|\nabla_\zeta \ln|z-\zeta|\right| \le \frac{1}{\delta}. \tag{4}$$

holds for $|z-\zeta| \ge \delta$.

Let $z = x+iy;\ \zeta = \eta + i\tau$ we have $\left(\ln|z-\zeta|\right)'_\eta = -\frac{x-\eta}{|z-\zeta|^2}; \left(\ln|z-\zeta|\right)'_\tau = -\frac{y-\tau}{|z-\zeta|^2}$.

From these estimates it is easy to obtain inequality

$$\nabla_\zeta \ln|z-\zeta| = \left(\left(\ln|z-\zeta|\right)'_\eta\right)^2 + \left(\left(\ln|z-\zeta|\right)'_\tau\right)^2 = = \frac{1}{|z-\zeta|} \le \frac{1}{\delta}.$$

Thus the inequality (4) is proved.

4. This is the most essential part of the proof. We claim that

$$\psi(z;z_0) \to 0 \qquad (z \to z_0).$$

Let δ be an arbitrary number. We assume that the inequality $|z-z_0| \le \delta$ holds. Consecutive application of the Minkowski inequality [4] and the inclusion $B(z_0;2\delta) \subset B(z;3\delta)$ yields

$$\psi(z;z_0) \le \left(\int\limits_{CB(z;3\delta)} |\ln|z-\zeta||^p\, d\gamma(\zeta)\right)^{1/p} + \left(\int\limits_{CB(z_0;2\delta)} |\ln|z-\zeta| - \ln|z_0-\zeta||^p\, d\gamma(\zeta)\right)^{1/p} +$$

$$+\left(\int\limits_{CB(z_0;2\delta)} |\ln|z_0-\zeta||^p\, d\gamma(\zeta)\right)^{1/p} = J_1 + J_2 + J_3.$$

If $\zeta \notin B(z_0;2\delta)$ and $|z-z_0| \le \delta$, then the inequalities $|w-\zeta| > \delta$ and

$\left|\nabla_\zeta \ln|z-\zeta|\right| \le \frac{1}{\delta}$ hold for any $w \in B(z_0;|z-z_0|)$.

The estimate of the gradient implies that

$$\left|\ln|z-\zeta| - \ln|z_0-\zeta|\right| \le \frac{1}{\delta}|z-z_0|;\ J_2 \le \frac{1}{\delta}\left(\gamma(C)\right)^{1/p}|z-z_0|.$$

From the hypothesis of the theorem we obtain that

$$J_1 \to 0; \quad J_3 \to 0 \quad (\delta \to 0),$$

which proves the required assertion.

The collection of assertions that we have obtained is contradictory. Thus the theorem is proved.

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